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ORIGINAL PAPER

Finite-time synchronization of nonidentical chaotic systems with multiple time-varying delays and bounded perturbations

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Abstract This paper is concerned with synchronization in a setting time for drive-response chaotic systems with multiple time-varying delays. The driving and response systems exhibit different dynamical behaviors with nonidentical delays and uncertain bounded external perturbations. Due to the time delays, existing finite-time stability theorem cannot be applied to the synchronization goal. By designing suitable controller and designing some Lyapunov–Krasovskii functionals, sufficient conditions guaranteeing the finite-time synchronization are derived without using existing finite-time stability theorem. Results of this paper extend most of existing ones which can only finite-timely synchronize coupled identical systems without delay. Numerical simulations demonstrate the effectiveness of the theoretical analysis.

Keywords Finite-time synchronization · Chaotic system · Time-varying delay · Perturbation

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1 Introduction

In the past decades, synchronization and control have been extensively studied due to the pioneering work of Pecora and Carroll in [1]. It is reported that synchronization has important applications in science and engineering such as secure communication, biological, and chemical reaction [2,3]. In the literature, there are many results concerning asymptotic synchronization [4–9]. Asymptotic synchronization means that the synchronization can only be realized as time goes to infinity. From practical point of view, one may expect that the synchronization is realized as fast as possible. In recent years, much efforts have been devoted to fast synchronization. An effective method for fast synchronization is to use finite-time technique. Moreover, finite-time technique has better robust and perturbation rejection ability than asymptotic technique [10]. Therefore, finite-time synchronization has attracted increasing attention in recent years.

Finite-time synchronization means that synchronization can be achieved in a setting time. In the literature, there are many results on finite-time synchronization. The authors in [11–14] considered finite-time synchronization of multi-agent systems. Finite-time synchronization of several classes of coupled chaotic systems was studied in [15–18]. However, most of existing finite-time synchronization results did not consider time delay. Time delays are an important internal factor of nonlinear systems and have heavy effect on the dynamics of nonlinear systems. For example, the quiescent state of neural networks can be transited to spiking, bursting, and chaotic state when there is time delay in the autapses [19,20]. It should be noted that most of existing results on synchronization in finite time are obtained by using the finite-time stability theorem in [21] or similar theorem in [22]. Unfortunately, it was reported in [23] that the finite-time stability theorems in [21] and [22] cannot be employed to study finite-time stability and synchronization of time-delay systems. Considering that time delay is unavoidable in practice, the authors in [24] considered finite-time issue for a special linear system with constant time delay. But the results in [24] are not usable in practice since it is extremely difficult to find a Lyapunov function satisfying the assumptions in [24]. Recently, Yang proposed a new control and analytical technique to study finite-time synchronization of neural networks with time delays in [25], but the driving and response systems are identical. Moreover, Yang did not consider external perturbations. A natural question is: whether the synchronization can still be realized in finite time when the driving and response systems are nonidentical with different uncertain external perturbations and time delays? As far as the authors' knowledge, finite-time synchronization of nonidentical chaotic systems with different uncertain external perturbations and delays has not been considered in the literature.

It is not always practical to assume that coupled systems are identical [26]. In [26] and [27], asymptotic synchronization of complex dynamical networks with nonidentical nodes has been studied. In [28] and [29], cluster synchronization in networks of coupled nonidentical dynamical systems has been investigated. In [30], partial synchronization of nonidentical chaotic systems via adaptive control. In [31] and [32], synchronization of nonidentical chaotic systems with time-varying delay was studied by impulsive control. Note that only asymptotic synchronization can be guaranteed in [26–32] for coupled nonidentical chaotic systems. Recently, by using the finite-time stability theorem in [22], the authors in [10] studied finite-time synchronization for a class of coupled nonidentical chaotic systems without delay. However, by the comments in [23], results in [10] cannot be directly extended to finite-time synchronization of nonidentical systems with time delays. This paper aims to solve this challenging problem.

External disturbances should also be considered in studying synchronization [33]. There are many papers in the literature concerning stability and synchronization of dynamical systems with bounded external perturbations, and at the same time, some effective control schemes have been proposed, such as robust control [34], H_∞ control [35], and sliding mode control [36]. However, to the best of our knowledge, most of published papers concerning synchronization of coupled systems with nonidentical perturbations can only drive the synchronization errors to some bounded areas around the origin. It is well known that one of the important applications of synchronization is secure communication, and the transmitted signals can only be recovered when the synchronization error has been driven to zero. Thus, it is of great importance to study finite-time synchronization of coupled systems with nonidentical external perturbations.

Based on the above discussions, this paper investigates the finite-time synchronization of nonidentical drive-response systems with different multiple time-varying delays and bounded external perturbations without using the finite-time stability theorem in [21]. A simple nonlinear controller is designed. By utilizing suitable Lyapunov–Krasovskii's functionals and some useful inequality techniques, several sufficient conditions are obtained to finite-timely synchronize the interest system. Moreover, the setting time for synchronization is also estimated. Numerical examples are given to show the effectiveness of our results.

The rest of this paper is organized as follows. In Sect. 2, models with uncertain external perturbations and multiple time-varying delays are presented, and some necessary definitions and assumptions are given. In Sect. 3, finite-time synchronization for the presented model is studied. In Sect. 4, several numerical simulations are given to demonstrate the effectiveness of the theoretical results. Sect. 5 comes to conclusion.

Notations: In this paper, the notations are quite standard. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $A = (a_{ij})_{n\times n}$ denotes a matrix of $n \times n$ real matrices. The superscript T denotes matrix or vector transposition. For a vector $x = (x_1, x_2, \ldots, x_n)^T \in R^n$, if $|x_i|, i = 1, 2, \ldots, n$ is bounded, then we say x is bounded. Let $\tau > 0$ and $C([-\tau, 0]; R^n)$ denote the family of continuous functions ω for $[-\tau, 0]$ to R^n with the uniform norm $\|\omega\| = \sup_{-\tau\leq\kappa\leq 0} |\omega(\kappa)|$. Sometimes, the arguments of a function or a matrix will be omitted in the analysis when no confusion can arise.

2 Preliminaries

Consider a general nonlinear system with multiple time-varying delays and bounded external perturbations as follows:

$$\dot{x}(t) = Ax(t) + f(x(t), x(t-\tau_1(t)), \dots, x(t-\tau_m(t))) + J(t, x(t)), \quad (1)$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in \mathbb{R}^n$ is the state of the system, $f(x(t), x(t-\tau_1(t)), \dots, x(t-\tau_m(t))) = (f_1(x(t), x(t-\tau_1(t)), \dots, x(t-\tau_m(t))), f_2(x(t), x(t-\tau_1(t)), \dots, x(t-\tau_m(t))), \dots, f_n(x(t), x(t-\tau_1(t)), \dots, x(t-\tau_m(t))))^T$, $J(t, x(t)) = (J_1(t, x(t)), J_2(t, x(t)), \dots, J_n(t, x(t)))^T \in \mathbb{R}^n$ is external perturbation, $\tau_1(t), \tau_2(t), \dots, \tau_m(t)$ are the internal multiple time-varying delays.

Consider (1) as the driving system, the response system is presented as follows:

$$\dot{y}(t) = \tilde{A}y(t) + \tilde{f}(y(t), y(t-\tilde{\tau}_1(t)), \dots, y(t-\tilde{\tau}_l(t))) + \tilde{J}(t, y(t)) + U(t), \quad (2)$$

where $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T \in \mathbb{R}^n$ is the state of the response system, $\tilde{f}(y(t), y(t-\tilde{\tau}_1(t)), \dots, y(t-\tilde{\tau}_l(t)))=(\tilde{f}_1(y(t), y(t-\tilde{\tau}_1(t)), \dots, y(t-\tilde{\tau}_l(t))), \tilde{f}_2(y(t), y(t-\tilde{\tau}_1(t)), \dots, y(t-\tilde{\tau}_l(t))), \dots, \tilde{f}_n(y(t), y(t-\tilde{\tau}_1(t)), \dots, y(t-\tilde{\tau}_l(t))))^T$, $\tilde{J}(t, y(t)) = (\tilde{J}_1(t, y(t)), \tilde{J}_2(t, y(t)), \dots, \tilde{J}_n(t, y(t)))^T \in \mathbb{R}^n$ is external perturbation, $\tilde{\tau}_1(t), \tilde{\tau}_2(t), \dots, \tilde{\tau}_l(t)$ are the internal multiple time-varying delays, and $U(t) = (u_1(t), u_2(t), \dots, u_n(t))^T$ is the controller to be designed.

The initial conditions of systems (1) and (2) are $x(s) = \varphi(s) = (\varphi_1(s), \varphi_2(s), \dots, \varphi_n(s))^T \in C([-\tau, 0], \mathbb{R}^n)$ and $y(s) = \phi(s) = (\phi_1(s), \phi_2(s), \dots, \phi_n(s))^T \in C([-\tau, 0], \mathbb{R}^n)$, respectively, where $\tau = \max\{\tau_1(t), \tau_2(t), \dots, \tau_m(t), \tilde{\tau}_1(t), \tilde{\tau}_2(t), \dots, \tilde{\tau}_l(t)\}$, $C([-\tau, 0], \mathbb{R}^n)$ denote the set of continuous vector-valued functions from $[-\tau, 0]$ to $\mathbb{R}^n$.

Let $e(t) = y(t)-x(t) = (e_1(t), e_2(t), \dots, e_n(t))^T$, $\tilde{f}(x(t), x(t-\tilde{\tau}_1(t)), \dots, x(t-\tilde{\tau}_l(t))) = \tilde{f}^x(t) = (\tilde{f}_1^x(t), \dots, \tilde{f}_n^x(t))^T$, $\tilde{f}(y(t), y(t-\tilde{\tau}_1(t)), \dots, y(t-\tilde{\tau}_l(t))) = \tilde{f}^y(t) = (\tilde{f}_1^y(t), \dots, \tilde{f}_n^y(t))^T$, $f(x(t), x(t-\tau_1(t)), \dots, x(t-\tau_m(t)))=f^x(t)=(f_1^x(t), \dots, f_n^x(t))^T$, $f(y(t), y(t-\tau_1(t)), \dots, y(t-\tau_m(t))) = f^y(t) = (f_1^y(t), \dots, f_n^y(t))^T$. The following error system can be obtained from (1) and (2).

$$\begin{aligned}\dot{e}(t) &= \tilde{A}e(t) + \tilde{g}(e(t), e(t-\tilde{\tau}_1(t)), \dots, e(t-\tilde{\tau}_l(t))) \\ &\quad + \tilde{A}x(t) + \tilde{f}(x(t), x(t-\tilde{\tau}_1(t)), \dots, x(t-\tilde{\tau}_l(t))) + \tilde{J}(t, y(t)) - Ax(t) \\ &\quad - f(x(t), x(t-\tau_1(t)), \dots, x(t-\tau_m(t))) \\ &\quad - J(t, x(t)) + U(t), \end{aligned} \quad (3)$$

or

$$\begin{aligned}\dot{e}(t) &= Ae(t) + g(e(t), e(t-\tau_1(t)), \dots, e(t-\tau_m(t))) \\ &\quad + \tilde{A}y(t) + \tilde{f}(y(t), y(t-\tilde{\tau}_1(t)), \dots, y(t-\tilde{\tau}_l(t))) + \tilde{J}(t, y(t)) - Ay(t) \\ &\quad - f(y(t), y(t-\tau_1(t)), \dots, y(t-\tau_m(t))) \\ &\quad - J(t, x(t)) + U(t), \end{aligned} \quad (4)$$

where $g(e(t), e(t-\tau_1(t)), \dots, e(t-\tau_m(t))) = (g_1(e(t), e(t-\tau_1(t)), \dots, e(t-\tau_m(t))), \dots, g_n(e(t), e(t-\tau_1(t)), \dots, e(t-\tau_m(t))))^T = f^y(t) - f^x(t)$, $\tilde{g}(e(t), e(t-\tilde{\tau}_1(t)), \dots, e(t-\tilde{\tau}_l(t))) = (\tilde{g}_1(e(t), e(t-\tilde{\tau}_1(t)), \dots, e(t-\tilde{\tau}_l(t))), \dots, \tilde{g}_n(e(t), e(t-\tilde{\tau}_1(t)), \dots, e(t-\tilde{\tau}_l(t))))^T = \tilde{f}^y(t) - \tilde{f}^x(t)$.

The initial condition of (3) and (4) is $e(s) = \phi(s) - \varphi(s) \in C([-\tau, 0], \mathbb{R}^n)$.

In this paper, the following assumptions are needed.

(H_1) There exist positive constants $\tau, \tilde{\tau}, \mu < 1, \tilde{\mu} < 1$ such that $0 < \tau_h(t) \leq \tau$, $0 < \tilde{\tau}_{\bar{h}}(t) \leq \tilde{\tau}$, $\dot{\tau}_h(t) \leq \mu$, $\dot{\tilde{\tau}}_{\bar{h}}(t) \leq \tilde{\mu}$, $\bar{h} = 1, 2, \dots, l$, $h = 1, 2, \dots, m$.

(H_2) There exist positive constants k_{ij}, $i, j = 1, \dots, n$ such that

$$|f_i^y(t) - f_i^x(t)| \leq \sum_{j=1}^n k_{ij} \left[|y_j(t) - x_j(t)| + \sum_{h=1}^m |y_j(t-\tau_h(t)) - x_j(t-\tau_h(t))| \right].$$

(H_3) There are positive constants $\tilde{k}_{ij}$, $i, j = 1, \dots, n$ such that

$$|\tilde{f}_i^y(t) - \tilde{f}_i^x(t)| \leq \sum_{j=1}^n \tilde{k}_{ij} \left[|y_j(t) - x_j(t)| + \sum_{h=1}^l |y_j(t-\tilde{\tau}_h(t)) - x_j(t-\tilde{\tau}_h(t))| \right].$$

(H_4) There are positive constants $M_i, \tilde{M}_i, H_i, L_i$ such that $|f_i^x(t)|, |f_i^y(t)| \leq M_i$, $|\tilde{f}_i^x(t)|, |\tilde{f}_i^y(t)| \leq \tilde{M}_i$, $|x_i(t)| \leq H_i$, $|y_i(t)| \leq L_i$, $i = 1, 2, \dots, n$.

(H_5) There exist constants Q_i and $\tilde{Q}_i$, such that $|J_i(t, x(t))| \leq Q_i$, $|\tilde{J}_i(t, y(t))| \leq \tilde{Q}_i$, $i = 1, 2, \dots, n$.

Remark 1 Conditions (H_2) and (H_3) are general, which contain almost all the well-known chaotic systems with or without delays such as Lur'e system, Lorenz system, Rässler system, Chen system, delayed Chua'a circuit, and neural network [10]. All these systems mentioned above can be described as $\dot{x} = f(t, x(t), x(t-\tau(t)))$, and there are positive constants $k_{ij} > 0, i, j = 1, 2, \cdots, n$ such that

$$|f_i(t, y(t), y(t-\tau(t))) - f_i(t, x(t), x(t-\tau(t)))| \leq \sum_{j=1}^{n} k_{ij}(|y_j(t) - x_j(t)| + |y_j(t-\tau(t)) - x_j(t-\tau(t))|) \tag{5}$$

for any $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ and $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T \in \mathbb{R}^n$, where $\tau(t)$ is time-varying delay. In this paper, the condition (5) is extended to multiple time delays.

Remark 2 The conditions (H_4) and (H_5) are general. It is well known that chaotic system has strange attractors, and there exists a bounded region containing all attractors of it such that every orbit of the system never leave them [32,33], which implies that the conditions (H_4) and (H_5) are easily satisfied for general chaotic systems.

Definition 1 The response system (2) is said to be synchronized with driving system (1) in finite time, if there exists a constant $t_1 > 0$ (t_1 depends on the initial state value of error system and time delay) such that $\|e(t_1)\|_1 = 0$ and $\|e(t)\|_1 \equiv 0$ for $t > t_1$, where $\|e(t)\|_1 = \sum_{i=1}^{n} |e_i(t)|$, t_1 is called the setting time.

3 Main results

In this section, by designing a suitable controller and using similar analytical techniques proposed in [25], several sufficient conditions are derived to guarantee the synchronization in a setting time. Moreover, the setting time is estimated theoretically.

Consider the following state feedback controller:

$$u_i = -\xi_i e_i(t) - \delta_i \mathrm{sgn}(e_i(t)), \quad i = 1, 2, \ldots, n, \tag{6}$$

where ξ_i, δ_i are positive constants to be determined, the symbol sgn denotes the standard sign function.

The following theorem is our main result.

Theorem 1 *Suppose that conditions (H_1), (H_3), (H_4), and (H_5) hold and the control gains ξ_i and δ_i in (6) satisfy the following inequalities:*

$$\xi_i \geq \sum_{j=1}^{n} \left[\tilde{a}_{ij} + \left(1 + \frac{1}{1-\tilde{\mu}} l\right) \tilde{k}_{ji} \right], \quad i = 1, 2, \ldots, n, \tag{7}$$

$$\delta_i > \sum_{j=1}^{n} |\tilde{a}_{ij} - a_{ij}| H_i + M_i + Q_i + \tilde{M}_i + \tilde{Q}_i, \quad i = 1, 2, \ldots, n. \tag{8}$$

Then system (2) can be synchronized with (1) in a setting time under controller (6). Moreover, the setting time is estimated as $t_1 \leq \frac{1}{\rho}(\sum_{i=1}^{n} |e_i(0)| + \frac{1}{1-\tilde{\mu}} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{h=1}^{l} \tilde{k}_{ij} \int_{-\tilde{\tau}}^{0} |e_j(s)| \mathrm{d}s)$, where $\rho = \min\{\rho_i, i = 1, 2, \ldots, n\}$, $\rho_i = \delta_i - (\sum_{j=1}^{n} |\tilde{a}_{ij} - a_{ij}| H_i + M_i + Q_i + \tilde{M}_i + \tilde{Q}_i)$.

Proof Define the following Lyapunov–Krasovskii functional candidate:

$$V(t) = V_1(t) + V_2(t), \tag{9}$$

where

$$V_1(t) = \sum_{i=1}^{n} |e_i(t)|,$$

$$V_2(t) = \frac{1}{1-\tilde{\mu}} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{h=1}^{l} \tilde{k}_{ij} \int_{t-\tilde{\tau}_h(t)}^{t} |e_j(s)| \mathrm{d}s.$$

Calculating the time derivative of $V_1(t)$ along the trajectories of the error system (3), it can be found that

$$\begin{aligned}
\dot{V}_1(t) &= \sum_{i=1}^{n} \mathrm{sgn}(e_i(t)) \dot{e}_i(t) \\
&= \sum_{i=1}^{n} \mathrm{sgn}(e_i(t)) \Bigg[\sum_{j=1}^{n} \tilde{a}_{ij} e_i(t) + \tilde{g}_i(t) + \sum_{j=1}^{n} \tilde{a}_{ij} x_i(t) + \tilde{f}_i^x(t) + \tilde{J}_i(t, y(t)) \\
&\quad - \sum_{j=1}^{n} a_{ij} x_i(t) - f_i^x(t) - J_i(t, x(t)) - \xi_i e_i(t) - \delta_i \mathrm{sgn}(e_i(t)) \Bigg] \\
&= \sum_{i=1}^{n} \left\{ \left(\sum_{j=1}^{n} \tilde{a}_{ij} - \xi_i \right) |e_i(t)| \right.
\end{aligned}$$

$$+\operatorname{sgn}(e_i(t))\left[\tilde{g}_i(t)+\sum_{j=1}^{n}(\tilde{a}_{ij}-a_{ij})x_i(t)+\tilde{f}_i^x(t)+\tilde{J}_i(t,y(t))-f_i^x(t)-J_i(t,x(t))\right]-\delta_i|\operatorname{sgn}(e_i(t))|\Bigg\}, \quad (10)$$

where $|\operatorname{sgn}(e_i(t))| = 1$ when $e_i(t) \neq 0$, while $|\operatorname{sgn}(e_i(t))| = 0$ when $e_i(t) = 0$.

It follows from (H_3) that

$$\begin{aligned}\operatorname{sgn}(e_i(t))\tilde{g}_i(t) &\le |\tilde{g}_i(t)| \\ &\le \sum_{j=1}^{n}\tilde{k}_{ij}\left[|y_j(t)-x_j(t)| + \sum_{h=1}^{l}|y_j(t-\tilde{\tau}_h(t))-x_j(t-\tilde{\tau}_h(t))|\right] \\ &= \sum_{j=1}^{n}\tilde{k}_{ij}\left[|e_j(t)|+\sum_{h=1}^{l}|e_j(t-\tilde{\tau}_h(t))|\right]. \end{aligned} \quad (11)$$

One has from (H_4) and (H_5) that

$$\begin{aligned}&\operatorname{sgn}(e_i(t))\left[\sum_{j=1}^{n}(\tilde{a}_{ij}-a_{ij})x_i(t) + \tilde{f}_i^x(t)+\tilde{J}_i(t,y(t))-f_i^x(t)-J_i(t,x(t))\right] \\ &\le \left[\sum_{j=1}^{n}|(\tilde{a}_{ij}-a_{ij})||x_i(t)|+|\tilde{f}_i^x(t)| + |\tilde{J}_i(t,y(t))|+|f_i^x(t)| + |J_i(t,x(t))|\right]|\operatorname{sgn}(e_i(t))| \\ &\le \left[\sum_{j=1}^{n}|\tilde{a}_{ij}-a_{ij}|H_i+M_i + Q_i+\tilde{M}_i+\tilde{Q}_i\right]|\operatorname{sgn}(e_i(t))|. \end{aligned} \quad (12)$$

Substituting inequalities (11) and (12) into (10) derives that

$$\begin{aligned}V_1(t) &\le \sum_{i=1}^{n}\Bigg\{\left(\sum_{j=1}^{n}\tilde{a}_{ij}-\xi_i\right)|e_i(t)| + \sum_{j=1}^{n}\tilde{k}_{ij}\left[|e_j(t)|+\sum_{h=1}^{l}|e_j(t-\tilde{\tau}_h(t))|\right] \\ &\quad + \left[\sum_{j=1}^{n}|\tilde{a}_{ij}-a_{ij}|H_i+M_i + Q_i+\tilde{M}_i+\tilde{Q}_i-\delta_i\right]|\operatorname{sgn}(e_i(t))|\Bigg\} \\ &= \sum_{i=1}^{n}\left[\sum_{j=1}^{n}(\tilde{a}_{ij}+\tilde{k}_{ji})-\xi_i\right]|e_i(t)| + \sum_{i=1}^{n}\left[\sum_{j=1}^{n}|\tilde{a}_{ij}-a_{ij}|H_i+M_i+Q_i + \tilde{M}_i+\tilde{Q}_i-\delta_i\right]|\operatorname{sgn}(e_i(t))| \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{h=1}^{l}\tilde{k}_{ij}|e_j(t-\tilde{\tau}_h(t))|. \end{aligned} \quad (13)$$

It is obtained from $V_2(t)$ that

$$\begin{aligned}\dot{V}_2(t) &= \frac{1}{1-\tilde{\mu}}\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{k}_{ij}\left[l|e_j(t)| - \sum_{h=1}^{l}(1-\dot{\tilde{\tau}}_h(t))|e_j(t-\tilde{\tau}_h(t))|\right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{k}_{ij}\left[\frac{1}{1-\tilde{\mu}}l|e_j(t)| - \sum_{h=1}^{l}\frac{1-\dot{\tilde{\tau}}_h(t)}{1-\tilde{\mu}}|e_j(t-\tilde{\tau}_h(t))|\right] \\ &\le \frac{1}{1-\tilde{\mu}}l\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{k}_{ij}|e_j(t)| - \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{h=1}^{l}\tilde{k}_{ij}|e_j(t-\tilde{\tau}_h(t))|, \end{aligned} \quad (14)$$

where (H_1) has been used.

By (13) and (14), one has

$$\begin{aligned}\dot{V}(t) &= \dot{V}_1(t)+\dot{V}_2(t) \\ &\le \sum_{i=1}^{n}\left[\sum_{j=1}^{n}\left(\tilde{a}_{ij}+\left(1+\frac{1}{1-\tilde{\mu}}l\right)\tilde{k}_{ji}\right)-\xi_i\right]|e_i(t)|\end{aligned}$$

$$+\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|\tilde{a}_{ij}-a_{ij}|H_i\right.$$

$$\left.+M_i+Q_i+\tilde{M}_i+\tilde{Q}_i-\delta_i\right)|\mathrm{sgn}(e_i(t))|$$

$$=\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\left(\tilde{a}_{ij}+\left(1+\frac{1}{1-\tilde{\mu}}l\right)\tilde{k}_{ji}\right)-\xi_i\right]|e_i(t)|$$

$$-\sum_{i=1}^{n}\rho_i\lambda_i, \tag{15}$$

where $\lambda_i=|\mathrm{sgn}(e_i(t))|$.

Substituting the conditions (7) and (8) into (15) yields

$$\dot{V}(t)\le-\sum_{i=1}^{n}\rho_i\lambda_i. \tag{16}$$

When $V(t)\neq 0$, it can be obtained from (9) that there exists at least one index $i\in\{1,2,\ldots,n\}$ such that $\lambda_i=1$. Therefore, when $V(t)\neq 0$, one has from (16) that

$$\dot{V}(t)\le-\rho_i\le-\rho<0. \tag{17}$$

Integrating both sides of the inequality (17) from 0 to t, one has

$$V(t)-V(0)\le-\rho t. \tag{18}$$

Now we prove that there exists an instant $t_1\in(0,+\infty)$ such that $V(t_1)=0$. Suppose that $V(t)>0$ for all $t>0$, then we have from (9) that $V_1(t)>0$ or $V_2(t)>0$. Since $V_2(t)>0$ for all $t>0$ implies $V_1(t)>0$, we only discuss the case $V_2(t)>0$. When $V_2(t)>0$, there exists $i_0\in\{1,2,\ldots,n\}$ such that $\int_{t-\tilde{\tau}_h(t)}^{t}|e_{i_0}(s)|\mathrm{d}s>0$. Then there exists $\tilde{t}\in(t-\tilde{\tau}_h(t),t)$ such that $|e_{i_0}(\tilde{t})|>0$. From the arbitrariness of $t>0$ one has that there exists $i_0\in\{1,2,\ldots,n\}$ such that $|e_{i_0}(t)|>0$ for all $t>0$, which means the inequality (18). Therefore, $\lim_{t\to+\infty}V(t)=-\infty$. This contradicts to the fact that $V(t)\ge 0$. Hence, there exist $t_1\in(0,+\infty)$ such that

$$\lim_{t\to t_1}V(t)=0 \ \text{ and } \ V(t)\equiv 0, \quad \text{for } \forall t\ge t_1. \tag{19}$$

By (18) and (19), one has

$$-V(0)\le-\rho t_1. \tag{20}$$

Considering (9) and (20), one can obtain that

$$t_1\le\frac{V(0)}{\rho}=\frac{1}{\rho}\left(\sum_{i=1}^{n}|e_i(0)|\right.$$

$$\left.+\frac{1}{1-\tilde{\mu}}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{h=1}^{l}\tilde{k}_{ij}\int_{-\tilde{\tau}}^{0}|e_j(s)|\mathrm{d}s\right). \tag{21}$$

This completes the proof. □

Remark 3 Note that the analytical method in the present paper is based on 1-norm and the key step in the proof of Theorem 1 is to obtain the inequality (17), which implies that the derivative of the Lyapunov functional is less than a negative constant before the realization of synchronization. However, the finite-time stability theorems in [21] and [22] are based on the inequality $\dot{V}(x)\le-\alpha V^{\eta}(x)$, where $\alpha>0$ and $0<\eta<1$ are constants, $\alpha_1(|x|)\le V(x)\le\alpha_2(|x|)$ with class-$\mathcal{K}$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$. The finite-time synchronization criteria in [11–18] are obtained by using the finite-time stability theorems in [21] and [22], and their Lyapunov functionals are of quadratic form (2-norm). If 2-norm-based Lyapunov functional are used, the inequality (17) cannot be derived.

When there is no delay in the driving and response systems, i.e., $\tau_h(t)=0$, $h=1,\ldots,m$ in the system (1) and $\tilde{\tau}_h(t)=0$, $h=1,\ldots,l$ in the system (2), the $V_1(t)=\sum_i^n|e_i(t)|$ can be taken as the Lyapunov function. By using the similar analysis as that in the proof of Theorem 1, one can easily derive the following corollary. Its proof is omitted.

Corollary 1 *Let $\tau_h(t)=0$, $h=1,\ldots,m$ and $\tilde{\tau}_h(t)=0$, $h=1,\cdots,l$. Suppose that conditions (H_3), (H_4), and (H_5) are satisfied, and the control gains ξ_i and δ_i in (6) satisfy the following inequalities:*

$$\xi_i\ge\sum_{j=1}^{n}(\tilde{a}_{ij}+\tilde{k}_{ji}),\quad i=1,2,\ldots,n, \tag{22}$$

$$\delta_i>\sum_{j=1}^{n}|\tilde{a}_{ij}-a_{ij}|H_i+M_i+Q_i+\tilde{M}_i+\tilde{Q}_i,$$

$$i=1,2,\ldots,n. \tag{23}$$

Then the systems (1) and (2) can be synchronized in a setting time under the controller (6). Moreover, the setting time is estimated as $t_1\le\frac{1}{\rho}\sum_{i=1}^{n}|e_i(0)|$, where $\rho=\min\{\rho_i,i=1,2,\cdots,n\}$.

The following theorem is abstained from the error system (4).

Theorem 2 *Suppose that the conditions* (H_1), (H_2), (H_4), *and* (H_5) *hold and the control gains* ξ_i *and* δ_i *in* (6) *satisfy the following inequalities:*

$$\xi_i \geq \sum_{j=1}^{n}\left[a_{ij}+\left(1+\frac{1}{1-\mu}m\right)k_{ji}\right], \quad i=1,2,\ldots,n, \tag{24}$$

$$\delta_i > \sum_{j=1}^{n}|\tilde{a}_{ij}-a_{ij}|L_i+M_i+Q_i+\tilde{M}_i+\tilde{Q}_i, \quad i=1,2,\ldots,n. \tag{25}$$

Then system (2) *can be synchronized with* (1) *in a setting time under the controller* (6). *Moreover, the setting time is estimated as* $t_1 \leq \frac{1}{\rho}(\sum_{i=1}^{n}|e_i(0)| + \frac{1}{1-\mu}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{h=1}^{m}k_{ij}\int_{-\tau}^{0}|e_j(s)|\mathrm{d}s)$, *where* $\rho = \min\{\rho_i, i=1,2,\ldots,n\}$, $\rho_i = \delta_i - (\sum_{j=1}^{n}|\tilde{a}_{ij} - a_{ij}|L_i + M_i + Q_i + \tilde{M}_i + \tilde{Q}_i)$.

Proof Define the following Lyapunov–Krasovskii functional candidate:

$$V(t) = V_1(t) + V_2(t), \tag{26}$$

where

$$V_1(t) = \sum_{i=1}^{n}|e_i(t)|,$$

$$V_2(t) = \frac{1}{1-\mu}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{h=1}^{m}k_{ij}\int_{t-\tau_h(t)}^{t}|e_j(s)|\mathrm{d}s.$$

Computing the derivative of $V_1(t)$ along trajectories of error system (4) yields

$$\begin{aligned}
\dot{V}_1(t) &= \sum_{i=1}^{n}\mathrm{sgn}(e_i(t))\dot{e}_i(t)\\
&= \sum_{i=1}^{n}\mathrm{sgn}(e_i(t))\Bigg[\sum_{j=1}^{n}a_{ij}e_i(t)\\
&\quad + g_i(t) + \sum_{j=1}^{n}\tilde{a}_{ij}y_i(t) + \tilde{f}_i^y(t) + \tilde{J}_i(t,y(t))\\
&\quad - \sum_{j=1}^{n}a_{ij}y_i(t) - f_i^y(t) - J_i(t,x(t))\\
&\quad - \xi_i e_i(t) - \delta_i\,\mathrm{sgn}(e_i(t))\Bigg]\\
&= \sum_{i=1}^{n}\Bigg\{\left(\sum_{j=1}^{n}a_{ij}-\xi_i\right)|e_i(t)|\\
&\quad + \mathrm{sgn}(e_i(t))\Bigg[g_i(t) + \sum_{j=1}^{n}(\tilde{a}_{ij}-a_{ij})y_i(t)\\
&\quad + \tilde{f}_i^y(t) + \tilde{J}_i(t,y(t)) - f_i^y(t)\\
&\quad - J_i(t,x(t))\Bigg] - \delta_i|\mathrm{sgn}(e_i(t))|\Bigg\}.
\end{aligned} \tag{27}$$

One has from (H_2) that

$$\begin{aligned}
\mathrm{sgn}(e_i(t))g_i(t) &\leq |g_i(t)|\\
&\leq \sum_{j=1}^{n}k_{ij}\left[|e_j(t)| + \sum_{h=1}^{m}|e_j(t-\tau_h(t))|\right].
\end{aligned} \tag{28}$$

Similar to the inequality (12), one has from (H_4) and (H_5) that

$$\begin{aligned}
&\mathrm{sgn}(e_i(t))\Bigg[\sum_{j=1}^{n}(\tilde{a}_{ij}-a_{ij})y_i(t) + \tilde{f}_i^y(t) + \tilde{J}_i(t,y(t))\\
&\quad - f_i^y(t) - J_i(t,x(t))\Bigg]\\
&\leq \Bigg[\sum_{j=1}^{n}|(\tilde{a}_{ij}-a_{ij})||y_i(t)| + |\tilde{f}_i^y(t)| + |\tilde{J}_i(t,y(t))|\\
&\quad + |f_i^y(t)| + |J_i(t,x(t))|\Bigg]|\mathrm{sgn}(e_i(t))|\\
&\leq \Bigg[\sum_{j=1}^{n}|\tilde{a}_{ij}-a_{ij}|L_i\\
&\quad + M_i + Q_i + \tilde{M}_i + \tilde{Q}_i\Bigg]|\mathrm{sgn}(e_i(t))|.
\end{aligned} \tag{29}$$

Substituting (28) and (29) into (27), one has

$$\begin{aligned}
V_1(t) &\leq \sum_{i=1}^{n}\left[\sum_{j=1}^{n}(a_{ij}+k_{ji})-\xi_i\right]|e_i(t)|\\
&\quad + \sum_{i=1}^{n}\Bigg[\sum_{j=1}^{n}|\tilde{a}_{ij}-a_{ij}|L_i + M_i
\end{aligned}$$

$$+ Q_i + \tilde{M}_i + \tilde{Q}_i - \delta_i \Bigg] |\mathrm{sgn}(e_i(t))| + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{h=1}^{m} k_{ij}|e_j(t-\tau_h(t))|. \tag{30}$$

It is followed by $V_2(t)$ and (H_1) that

$$\begin{aligned}\dot{V}_2(t) &= \frac{1}{1-\mu}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}\Bigg[m|e_j(t)| - \sum_{h=1}^{m}(1-\dot{\tau}_h(t))|e_j(t-\tau_h(t))| \Bigg] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}\Bigg[\frac{1}{1-\mu} m|e_j(t)| - \sum_{h=1}^{m}\frac{1-\dot{\tau}_h(t)}{1-\mu}|e_j(t-\tau_h(t))| \Bigg] \\ &\le \frac{1}{1-\mu} m \sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}|e_j(t)| - \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{h=1}^{m} k_{ij}|e_j(t-\tau_h(t))|. \end{aligned} \tag{31}$$

Considering (30) and (31), one derives that

$$\begin{aligned}\dot{V}(t) &= \dot{V}_1(t) + \dot{V}_2(t) \\ &\le \sum_{i=1}^{n}\Bigg[\sum_{j=1}^{n}\left(a_{ij} + \left(1+\frac{1}{1-\mu}m\right)k_{ji}\right) - \xi_i\Bigg] \\ &\quad \times |e_i(t)| + \sum_{i=1}^{n}\Bigg(\sum_{j=1}^{n}|\tilde{a}_{ij} - a_{ij}|L_i \\ &\quad + M_i + Q_i + \tilde{M}_i + \tilde{Q}_i - \delta_i\Bigg)|\mathrm{sgn}(e_i(t))| \\ &= \sum_{i=1}^{n}\Bigg[\sum_{j=1}^{n}\left(a_{ij} + \left(1+\frac{1}{1-\mu}m\right)k_{ji}\right) - \xi_i\Bigg] \\ &\quad \times |e_i(t)| - \sum_{i=1}^{n}\rho_i\lambda_i, \end{aligned} \tag{32}$$

where $\lambda_i = |\mathrm{sgn}(e_i(t))|$.

Substituting the conditions (24) and (25) into (32) produces the following inequality:

$$\dot{V}(t) \le -\rho. \tag{33}$$

The rest part is the same as that in the proof of Theorem 1. This completes the proof. □

The following corollary can be derived from Theorem 2.

Corollary 2 *Let $\tau_h(t) = 0$, $h = 1, \ldots, m$ and $\tilde{\tau}_h(t) = 0$, $h = 1, \ldots, l$. Suppose that conditions (H_2), (H_4), and (H_5) are satisfied and the control gains ξ_i and δ_i in (6) satisfy the following inequalities:*

$$\xi_i \ge \sum_{j=1}^{n}(a_{ij} + k_{ji}), \quad i = 1, 2, \ldots, n, \tag{34}$$

$$\delta_i > \sum_{j=1}^{n}|\tilde{a}_{ij} - a_{ij}|L_i + M_i + Q_i + \tilde{M}_i + \tilde{Q}_i, \quad i = 1, 2, \ldots, n. \tag{35}$$

Then the systems (1) and (2) can be synchronized in a setting time under the controller (6). Moreover, the setting time is estimated as $t_1 \le \frac{1}{\rho}\sum_{i=1}^{n}|e_i(0)|$, where $\rho = \min\{\rho_i, i = 1, 2, \cdots, n\}$.

Remark 4 Based on the error systems (3) and (4), Theorems 1 and 2 are obtained respectively. All the synchronization criteria in Theorems 1 and 2, and Corollaries 1, 2 are applicable to finite-time synchronization control of nonidentical systems. In real applications, operators should select less conservative synchronization criteria from Theorems 1 and 2, and Corollaries 1, 2 according to the practical situation.

When the drive-response systems (1) and (2) are identical and do not have external perturbations, i.e., $A = \tilde{A}$, $f^x(t) = \tilde{f}^x(t)$, $J(t, x(t)) = \tilde{J}(t, y(t)) = 0$, the error systems (3) and (4) turn out to be

$$\dot{e}(t) = Ae(t) + g(e(t), e(t-\tau_1(t)), \ldots, e(t-\tau_m(t))) + U(t). \tag{36}$$

From Theorem 2, one can obtain the following corollary.

Corollary 3 *Suppose that $A = \tilde{A}$, $f^x(t) = \tilde{f}^x(t)$, $J(t, x(t)) = \tilde{J}(t, y(t)) = 0$, and (H_1), (H_2) hold. If the control gains ξ_i and δ_i in (6) satisfy*

$$\xi_i \ge \sum_{j=1}^{n}\left[a_{ij} + \left(1+\frac{1}{1-\mu}m\right)k_{ji}\right], \quad i = 1, 2, \ldots, n, \tag{37}$$

$$\delta_i > 0, \quad i = 1, 2, \ldots, n. \tag{38}$$

Then system (2) can be synchronized with (1) in a setting time under the controller (6). Moreover, the setting time is estimated as $t_1 \le \frac{1}{\delta}(\sum_{i=1}^{n}|e_i(0)| +$

$\frac{1}{1-\mu}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{h=1}^{m} k_{ij}\int_{-\tau}^{0}|e_j(s)|\mathrm{d}s)$, *where* $\delta = \min\{\delta_i, i = 1, 2, \ldots, n\}$.

Proof By taking $L_i = 0$, $M_i = 0$, $Q_i = 0$, $\tilde{M}_i = 0$, and $\tilde{Q}_i = 0$, the proof is the same as that in the proof of Theorem 2. This completes the proof. □

Remark 5 As far as the authors' knowledge, few published papers study finite-time synchronization of nonidentical systems except [10]. However, the results in [10] cannot be extended to time delay since they are obtained under the framework finite-time stability theorem in [22]. Moreover, [10] did not consider external disturbances. Recently, Yang studied finite-time synchronization of a class of neural networks with delays in [25], but the driving and response systems in [25] were identical and did not consider external perturbations. In this paper, the driving and response systems are nonidentical with different time delays and subject to bounded uncertain nonidentical external perturbations. Hence the model considered in the present paper is very general and improve the corresponding results in [10] and [25] to some extent.

4 Numerical examples

In this section, three numerical examples are given to illustrate the effectiveness of the theoretical results. Specifically, Examples 1, 2, and 3 are given to verify Theorems 1, 2, and Corollary 2, respectively. Moreover, the setting times for each example are also given.

Example 1 Consider two nonidentical neural networks with different multiple time-varying delays and uncertain nonlinear external perturbations in drive-response configuration. The driving system is represented as follows:

$$\dot{x}(t) = Ax(t) + f^x(t) + J(t, x(t)), \tag{39}$$

where $x(t) = (x_1(t), x_2(t))^T$, $f^x(t) = f(x(t), x(t - \tau_1(t)), x(t-\tau_2(t))) = (\tanh(x_1(t)) - 0.3\tanh(x_2(t)) + 1.4\tanh(x_1(t - \tau_1(t))) + 0.1\tanh(x_2(t - \tau_2(t))), -8\tanh(x_1(t)) + 5\tanh(x_2(t)) + 0.3\tanh(x_1(t-\tau_1(t))) - 8\tanh(x_2(t - \tau_2(t)))^T$, $\tau_1(t) = 0.1|\sin(t)|$, $\tau_2(t) = 0.8|\sin(t)|$, and

$$A = \begin{pmatrix} -3 & 2 \\ 1 & -1 \end{pmatrix}, \quad J(t, x(t)) = \begin{pmatrix} 0.1\cos(t) \\ 0.6 + 0.3\sin(t) \end{pmatrix}.$$

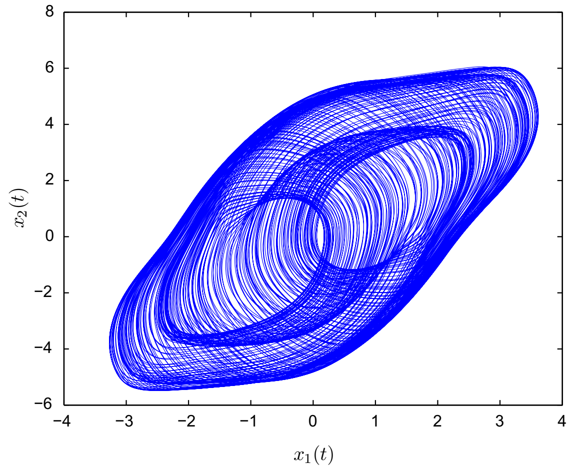

Fig. 1 Trajectories of $x(t)$ of (39) with initial conditions $x(0) = (1, 3)^T$

The response system is described by

$$\dot{y}(t) = \tilde{A}y(t) + \tilde{f}^y(t) + \tilde{J}(t, y(t)) + U(t), \tag{40}$$

where $y(t) = (y_1(t), y_2(t))^T$, $\tilde{f}^y(t) = \tilde{f}(y(t), y(t - \tilde{\tau}_1(t))) = (2.3\tanh(y_1(t)) + 0.3\tanh(y_2(t)) - \tanh(y_1(t - \tilde{\tau}_1(t))), 4\tanh(y_1(t)) + 4\tanh(y_2(t)) + 0.6\tanh(y_1(t - \tilde{\tau}_1(t))))^T$, $\tilde{\tau}_1(t) = 0.4 * |\sin(t)|$, and

$$\tilde{A} = \begin{pmatrix} 0.2 & -1 \\ 2 & -3 \end{pmatrix}, \quad \tilde{J}(t, y(t)) = \begin{pmatrix} 0.3\sin(t) \\ 0.2\cos(t) \end{pmatrix}.$$

Figure 1 presents the chaotic trajectory of (39) with the initial value $x(0) = (1, 3)^T$, and the states are bound with $|x_1| \le 3.6169 = H_1$, $|x_2| \le 6.0533 = H_2$. Figure 2 presents the chaotic trajectory of (40) with $U(t) = 0$ and the initial value $y(0) = (2, 1)^T$.

It is easy to get that $\tau = 0.8$ and $\mu = 0.8$, $\tilde{\tau} = 0.4$ and $\mu = 0.4$, then the condition (H_1) holds. One obtains that $|\tilde{f}_1^y(t) - \tilde{f}_1^x(t)| \le 2.3|e_1(t)| + 0.3|e_2(t)| + |e_1(t - \tilde{\tau}_1)|$ and $|\tilde{f}_2^y(t) - \tilde{f}_2^x(t)| \le 2.3|e_1(t)| + 0.3|e_2(t)| + |e_1(t - \tilde{\tau}_1)|$, thus the condition (H_3) is satisfied with $\tilde{k}_{11} = 2.3$, $\tilde{k}_{12} = 0.3$, $\tilde{k}_{21} = 4$, $\tilde{k}_{22} = 4$. Moreover, $f_1^x(t) \le |\tanh(x_1(t))| + 0.3|\tanh(x_2(t))| + 1.4|\tanh(x_1(t - \tau_1(t)))| + 0.1|\tanh(x_2(t - \tau_2(t)))| \le 1 + 0.3 + 1.4 + 0.1 = 2.8 = M_1$, $f_2^x(t) \le 8|\tanh(x_1(t))| + 5|\tanh(x_2(t))| + 0.3|\tanh(x_1(t - \tau_1(t)))| + 8|\tanh(x_2(t - \tau_2(t))| \le 8 + 5 + 0.3 + 8 = 21.3 = M_2$, $\tilde{f}_1^x(t) \le 2.3|\tanh(x_1(t))| + 0.3|\tanh(x_2(t))| + |\tanh(x_1(t - \tilde{\tau}_1(t)))| \le 2.3 + 0.3 + 1 = 3.6 = \tilde{M}_1$, $\tilde{f}_2^x(t) \le 4|\tanh(x_1(t))| +$

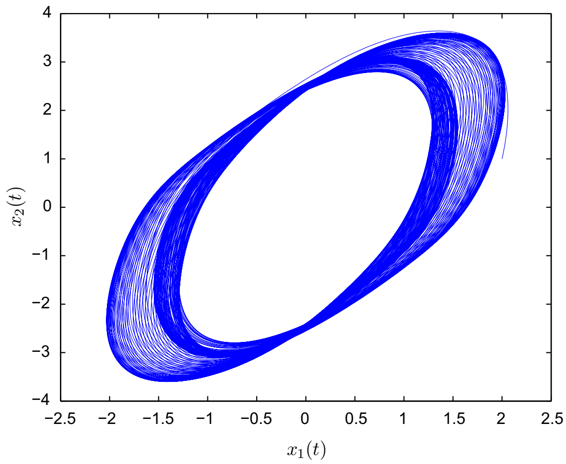

Fig. 2 Trajectories of $x(t)$ of (40) with initial conditions $y(0) = (2, 1)^T$

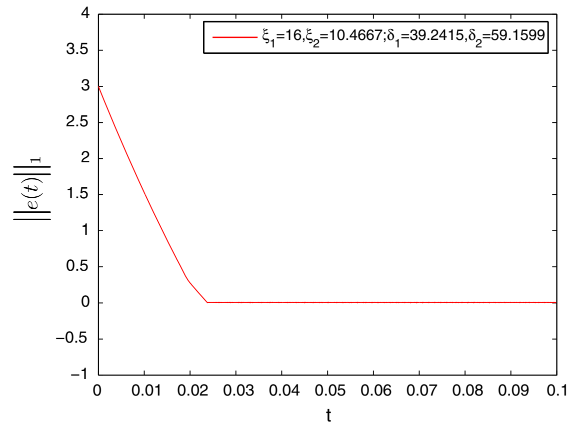

Fig. 3 Time evolution $\|e(t)\|_1$ with $\xi_1 = 16$, $\xi_2 = 10.4667$, and $\delta_1 = 39.2415$, $\delta_2 = 59.1599$

$4|\tanh(x_2(t))| + 0.6|\tanh(x_1(t - \tilde{\tau}_1(t)))| \le 4 + 4 + 0.6 = 8.6 = \tilde{M}_2$, and $J_1(t, x(t)) \le 0.1 = Q_1$, $J_2(t, x(t)) \le 0.9 = Q_2$, $\tilde{J}_1(t, y(t)) \le 0.3 = \tilde{Q}_1$, $\tilde{J}_2(t, y(t)) \le 0.2 = \tilde{Q}_2$. Hence, the conditions (H_4) and (H_5) are satisfied.

Take $\xi_1 = \sum_{j=1}^{n}\left[\tilde{a}_{1j} + (1 + \frac{1}{1-\tilde{\mu}}l)\tilde{k}_{j1}\right] = 16$, $\xi_2 = \sum_{j=1}^{n}\left[\tilde{a}_{2j} + (1 + \frac{1}{1-\tilde{\mu}}l)\tilde{k}_{j2}\right] = 10.4667$, and $\delta_1 > \sum_{j=1}^{n}|\tilde{a}_{1j} - a_{1j}|H_1 + M_1 + Q_1 + \tilde{M}_1 + \tilde{Q}_1 = 29.2415$, $\delta_2 > \sum_{j=1}^{n}|\tilde{a}_{2j} - a_{2j}|H_2 + M_2 + Q_2 + \tilde{M}_2 + \tilde{Q}_2 = 49.1599$, then the neural networks (39) and (40) can realize finite-time synchronization under the controller (6) according to Theorem 1. Now we take $\delta_1 = 39.2415$, $\delta_1 = 59.1599$. By simple computation, one has $\rho_1 = \rho_2 = 10$. Moreover, the setting time is $t_1 = 2.42$.

In the simulations, the Euler scheme is used and the step-length is set as 0.0001. The initial values of the systems (39) and (40) as those in Figs. 1 and 2, respectively, we obtain the time evolution of $\|e(t)\|_1$ showing in Fig. 3, from which one can see that the synchronization is achieved before the setting time $t_1 = 2.42$. Therefore, the effectiveness of Theorem 1 is verified.

Example 2 Consider the time-delay Rössler system with external perturbations as the driving system and the time-delay Lorenz system with external perturbations as the response system. The time-delay Rössler system with external perturbations is described as:

$$\dot{x}(t) = Ax(t) + f^x(t) + J(t, x(t)), \tag{41}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^T$, $f^x(t) = f(x(t), x(t - \tau_1(t)), x(t - \tau_2(t))) = (0, 0, 0.2 + x_1(t - \tau_1(t))x_3(t - \tau_2(t)))^T$, $\tau_1(t) = 0.05$, $\tau_2(t) = 0.1$, and

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0.38 & 0 \\ 0 & 0 & -5.7 \end{pmatrix}, \quad J(t, x(t)) = \begin{pmatrix} 0.1 \\ 0.3 \\ 0.5 \end{pmatrix}.$$

The time-delay Lorenz system with external perturbations is described as:

$$\dot{x}(t) = \tilde{A}y(t) + \tilde{f}^y(t) + \tilde{J}(t, y(t)) + U(t), \tag{42}$$

where $y(t) = (y_1(t), y_2(t), y_3(t))^T$, $\tilde{f}^y(t) = \tilde{f}(y(t), y(t - \tilde{\tau}_1(t)), y(t - \tilde{\tau}_2(t))) = (0, y_1(t)y_3(t - \tilde{\tau}_1(t)), y_1(t)y_2(t - \tilde{\tau}_2(t)))^T$, $\tilde{\tau}_1(t) = 0.1$, $\tilde{\tau}_2(t) = 0.04$, and

$$\tilde{A} = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & 8/3 \end{pmatrix}, \quad \tilde{J} = \begin{pmatrix} 0.8 \\ 1 \\ 1.3 \end{pmatrix}.$$

Choosing the initial value as $x(t) = (1, 1, 1)^T, \forall t \in [-0.1, 0]$, the chaotic trajectory of (41) is shown in Fig. 4. When take $u(t) = 0$ and the initial value is chosen as $y(t) = (3, 2, 4)^T, \forall t \in [-0.1, 0]$, the chaotic trajectory of (42) is presented in Fig. 5.

As the same solution procedure of Example 1, we get that $\tau = 0.1$, $\mu = 0$, $\tilde{\tau} = 0.1$, $\tilde{\mu} = 0$, $L = (27.1251, 47.7637, 108.$
$0509)^T$, $M_i = (0, 0, 529.6247)^T$, $Q = (0.1, 0.3, 0.5)^T$, $\tilde{M}_i = (0, 223.4931, 189.6356)^T$, $\tilde{Q} = (0.8, 1, 1.3)^T$, and

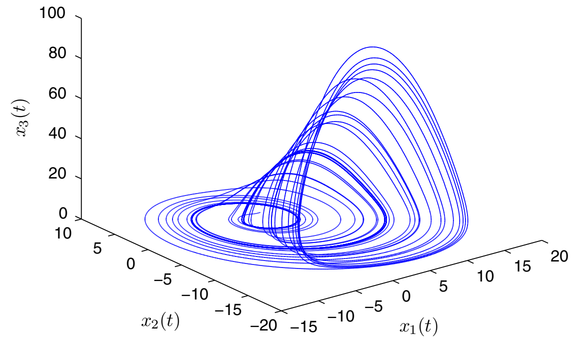

Fig. 4 Trajectories of $x(t)$ of (41) with initial conditions $x(t) = (1, 1, 1)^T, t = -0.1$

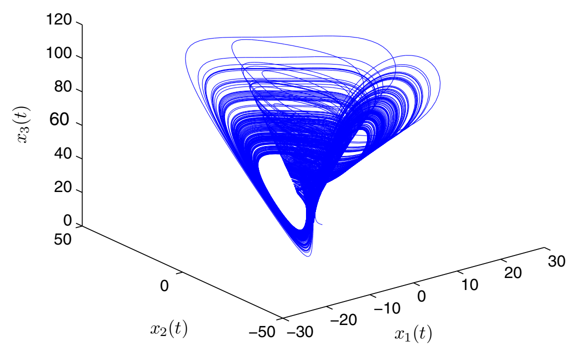

Fig. 5 Trajectories of $x(t)$ of (42) with initial conditions $x(t) = (3, 2, 4)^T, t = -0.1$

$$K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 64.9978 & 0 & 13.3675 \end{pmatrix}.$$

Take $\xi_1 = \sum_{j=1}^{n}\left[a_{1j} + (1 + \frac{1}{1-\mu}m)k_{j1}\right] = 192.9934$, $\xi_2 = \sum_{j=1}^{n}\left[a_{2j} + (1 + \frac{1}{1-\mu}m)k_{j2}\right] = 1.3800$, and $\xi_3 = \sum_{j=1}^{n}\left[a_{3j} + (1 + \frac{1}{1-\mu}m)k_{j3}\right] = 34.4025$. Choosing $\delta_1 = 607.65 > \sum_{j=1}^{n}|\tilde{a}_{1j} - a_{1j}|L_1 + M_1 + Q_1 + \tilde{M}_1 + \tilde{Q}_1 = 597.65$, $\delta_2 = 668.03 > \sum_{j=1}^{n}|\tilde{a}_{2j} - a_{2j}|L_2 + M_2 + Q_2 + \tilde{M}_2 + \tilde{Q}_2 = 658.03$, and $\delta_3 = 1635.10 > \sum_{j=1}^{n}|\tilde{a}_{3j} - a_{3j}|L_3 + M_3 + Q_3 + \tilde{M}_3 + \tilde{Q}_3 = 1625.10$, it gets that $\rho_1 = 10$, $\rho_2 = 10$, and $\rho_3 = 10$. Therefore, all the conditions of Theorem 2 hold, then the system (42) can be synchronized with (41) in finite time.

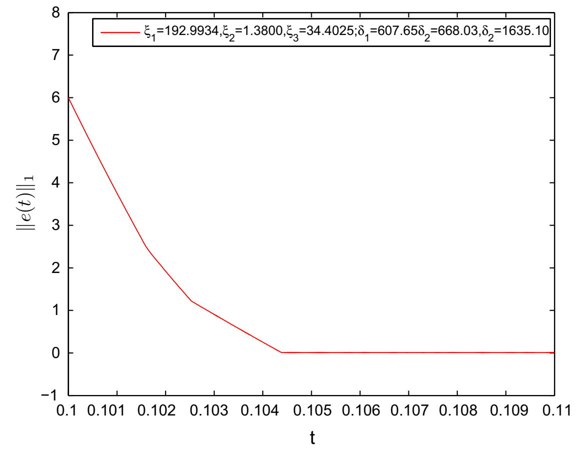

Fig. 6 Time evolution $\|e(t)\|_1$ with $\xi_1 = 192.9934$, $\xi_2 = 1.3800$, $\xi_3 = 34.4025$, and $\delta_1 = 607.65$, $\delta_2 = 668.03$, $\delta_3 = 1635.10$

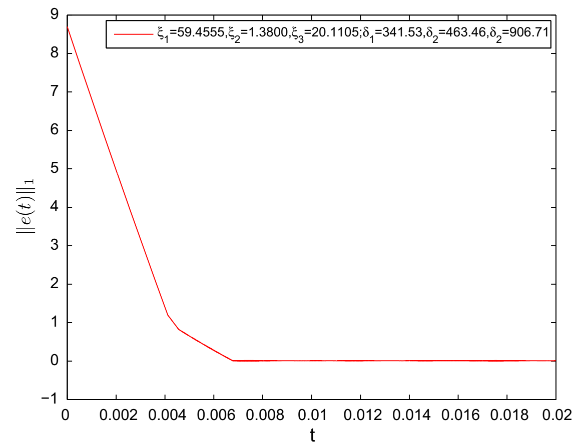

Fig. 7 Time evolution $\|e(t)\|_1$ with $\xi_1 = 59.4555$, $\xi_2 = 1.3800$, $\xi_3 = 20.1105$, and $\delta_1 = 341.53$, $\delta_2 = 463.46$, $\delta_3 = 906.71$

Taking the initial value of the systems (41) and (42) the same as that in Figs. 4 and 5, respectively, Fig. 6 describes the trajectories of the error states, which indicates that the synchronization is realized before the setting time $t_1 = 0.9578$ estimated by Theorem 2. Hence, Theorem 2 is verified.

Example 3 Consider system (41) without delay as the driving system and system (42) without delay as the response system. The initial value of the drive system is set as $x(0) = (0.1, 0.1, 0.1)^T$ and the initial value of the response system is set as $y(0) = (3, 2, 4)^T$.

It is easy to verify that all the conditions of Corollary 2 are satisfied if we take $\xi_1 = \sum_{j=1}^{n}(a_{1j} + k_{j1}) =$

59.4555, $\xi_2 = \sum_{j=1}^{n}(a_{2j} + k_{j2}) = 1.3800$, $\xi_3 = \sum_{j=1}^{n}(a_{ij} + k_{ji}) = 20.1105$, and $\delta_1 = 341.53 > \sum_{j=1}^{n} |\tilde{a}_{1j} - a_{1j}|L_1 + M_1 + Q_1 + \tilde{M}_1 + \tilde{Q}_1 = 331.53$, $\delta_2 = 463.46 > \sum_{j=1}^{n} |\tilde{a}_{2j} - a_{2j}|L_2 + M_2 + Q_2 + \tilde{M}_2 + \tilde{Q}_2 = 453.46$, $\delta_3 = 906.71 > \sum_{j=1}^{n} |\tilde{a}_{3j} - a_{3j}|L_3 + M_3 + Q_3 + \tilde{M}_3 + \tilde{Q}_3 = 896.71$. According to Corollary 2, we obtain that the setting time $t_1 = 0.87$. Fig. 7 shows the trajectories of the error states. In Fig. 7, the actual synchronization time is 0.069, which is shorter than the setting time.

5 Conclusions

In this paper, finite-time synchronization for a class of nonidentical drive-response chaotic systems with multiple time-varying delays and bounded external perturbations has been studied. By designing suitable controllers and Lyapunov functionals, several sufficient conditions have been obtained to guarantee that the coupled systems can be finite-timely synchronized. The setting time is also given. Models considered in this paper are general and the results extend existing ones to some extent. Three numerical examples have been given to verify the effectiveness of the theoretical results.

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